PEI5I103 DIGITAL SIGNAL PROCESSING

Module-I(10hours)

Discrete Time System

Basic Discrete Time Signals and their classifications, Discrete times systems and their classifications, Stability of discrete time system, Analysis and response (convolution sum) of discrete - time linear LTI system, Recursive and Non-recursive discrete time system, impulse response of LTI system, Correlation of discrete time Signal

The Z-Transform and Its Application to the Analysis of LTI Systems:

The Z-Transform: The Direct Z-Transform, The Inverse Z-Transform; Properties of the Z-Transform; Inversion of the Z-Transforms: The Inversion of the Z-Transform by Power Series Expansion, The Inversion of the Z-Transform by Partial-Fraction Expansion; Analysis of Linear Time-Invariant Systems in the z-Domain: Response of Systems with rational System Functions, Transient and Steady-State Responses, Causality and Stability, Pole-ZeroCancellations.

Module-II (15hours)

The Discrete Fourier Transform: Its Properties and Applications

Frequency Domain Sampling: Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals, The Discrete Fourier Transform, The DFT as a Linear Transformation, Relationship of the DFT to other Transforms; Properties of the DFT: Periodicity, Linearity, and Symmetry Properties, Multiplication of Two DFTs and Circular Convolution, Additional DFT Properties; Linear Filtering Methods Based on the DFT: Use of the DFT in Linear Filtering, Filtering of Long Data Sequences; Frequency Analysis of Signals using the DFT; The Discrete Cosine Transform: Forward DCT, Inverse DCT, DCT as an Orthogonal Transform.

Efficient Computation of the DFT: Fast Fourier Transform Algorithms

Efficient Computation of the DFT: FFT Algorithms: Direct Computation of the DFT, Radix-2 FFT Algorithms: Decimation-In-Time (DIT), Decimation-In-Time (DIF); Applications of FFT Algorithms: Efficient Computation of the DFT of two Real Sequences, Efficient Computation of the DFT a 2N-Point Real Sequence, Use of the FFT Algorithm in Linear Filtering and Correlation.

Module-III **(10hours)**

Structure and implementation of FIR and IIR filter:

Structure for the Realization of Discrete-Time Systems, Structure for FIR Systems: Direct- Form Structure, Cascade-Form Structures, Frequency-Sampling Structures; Structure for IIR Systems: Direct-Form Structures, Signal Flow Graphs and Transposed Structures, Cascade-Form Structures, Parallel-Form Structures.

Design of Digital Filters:

General Considerations: Causality and Its Implications, Characteristics of Practical Frequency-Selective Filters; Design of FIR Filters: Symmetric and Antisymmetric FIR Filters, Design of Linear-Phase FIR Filters by using Windows, Design of Linear-Phase FIR Filters by the Frequency-Sampling Method;

Design of IIR Filters from Analog Filters: IIR Filter Design by Impulse Invariance, IIR Filter Design by the Bilinear Transformation.

Basic adaptive filter: System modeling and Identifications using adaptive filter

Text Books

1. *Digital Signal Processing – Principles, Algorithms and Applications by J. G. Proakis and D. G. Manolakis, 4th Edition,Pearson.*
2. *Digital Signal Processing – S. Salivahan, A. Vallavraj and C. Gnanapriya, Tata McGrawHill.*
3. *Digital Signal Processing: a Computer-Based Approach – Sanjit K. Mitra, Tata McGraw Hill.*

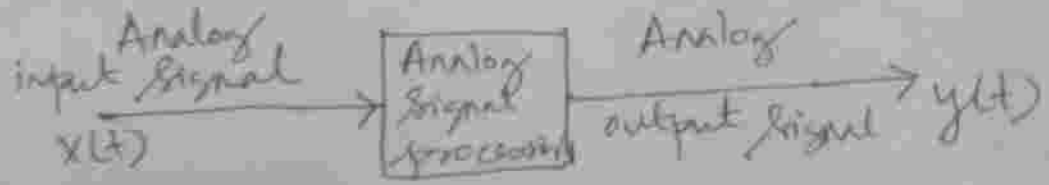
DSP

Introduction :-

→ A Signal is defined as any physical quantity that varies with time, Space, or any other independent variable. Ex - $x_1(t) = 2t$ and $x_2(t) = t^2$.

→ A System is defined as a physical device that performs an operation on a Signal.

→ Signal processing is any operation that changes the characteristics of a Signal. These characteristics means :- the amplitude, Shape, phase and frequency content of the Signal.



Analog Signal processing System

Digital Signal processing

→ The Source of the input Signal is from a transducer or a communication Signal, the Signal be an EEG or an ECG.

→ Input Signal is applied to anti-Aliasing filter used to remove the high frequency noise and to band limit the Signal.

→ The S/H device provides the input to the ADC and will be required if the input signal must remain relatively constant during the conversion of the Analog Signal to digital format.

Advantages of DSP

→ Greater Accuracy – Analog filter affects the accuracy where as DSP provides Superior control of accuracy.

→ cheaper.

→ Ease of data Storage.

→ Implementation of Sophisticated alogorithm.

→ flexibility in configuration.

→ Application to VLF Signals :- The very low frequency signals such that occurs in Seismic application can be easily processed using a DSP compared to ASP.

→ Time Sharing:- DSP allows the Sharing of a given processor among a number of Signals by time Sharing thus Reducing the cost of processing a Signal.

Limitations

→ System complexity :- DSP is more complex than ASP because of the devices such as A/D & D/A converters and their associated filters.

→ Bandwidth limited by Sampling rate.

→ Power consumption:- DSP chip containing over 4 lakh Resistors dissipates more power.

Application of DSP

→ Telecommunication :- Echo cancellation in telephone networks, Telephone dialling application. Modems, Line repeater, FAX, cellular phone.

→ Consumer electronics :- FM Stereo, Sound applications, Digital Audio/TV.

→ Instrumentation and control :- Servo control, Robot control and process control.

→ Image processing :- Image compression, image enhancement, Image Analysis & Recognition

→ Medicine :- CT, x-ray Scanning

→ Speech processing :- Speech analysis methods are used in automatic speech Recognition, Speaker verification and Speaker identification.

→ Seismology :- geophysical exploration for oil and gas, detection of underground nuclear explosion and earthquake monitoring.

→ Military :- Radar / Sonar Signal processing Navigation, Secure communications.

Classifications of Signals

In general (two types)

(a) Continuous-time Signal – A Signal that varies continuously with time. It can be represented as $x(t)$, where time 't' represents the independent variable. Ex – Sinusoidal Signal.

(b) Discrete-time Signal:– A Signal that has values only at discrete instants of time

→ one way obtaining DTS is, Sampling a continuous-time Signal at regular intervals.

DTS → $x(nT)$ where $n = 0, 1, 2, \ldots$ n are

T is the time interval between consecutive Signal values.

→ DTS is discrete in time but continuous in Amplitude

* DTS (Discrete time Signal) is a digital Signal which is discrete in both time and Amplitude. It is called digital because their samples are represented by numbers or digits.

* Signal Representation

1. Functional Representation.
2. Tabular Representation
3. Sequence Representation.

→ An infinite duration Sequence with time origin ($n=0$) indicated by the Symbol ↑ is represented as

$$x(n) = \{ \ldots 0, \underset{\uparrow}{1}, 1.5, 2, 0.5, 0.6, 1.0 \ldots \}$$

(3)

A finite duration sequence can be represented

$$x(n) = \{-2, -1, \underset{\uparrow}{2}, 1, 3, 5\}$$

→ A finite duration sequence that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as

$$x(n) = \{\underset{\uparrow}{0}, 2, 4, 2\}$$

Standard Discrete-time Signals

(1) Unit Sample Sequence.
(2) Unit Step Sequence.
(3) Unit Ramp ".

$\delta(n) = 1$ for $n = 0$
$= 0$ for $n \neq 0$.

→ $u(n) = 1$ for $n \geq 0$.
$= 0$ for $n < 0$.

→ $u_r(n) = n$ for $n \geq 0$.
$= 0$ for $n < 0$.

Expontetial Sequence

$x(n) = a^n$ for all n.

Periodic and Aperiodic Signals ④

(For Both continuous and Discrete time Signal)

→ For Discrete time Signal, the condition for periodicity can be written as

$$x(n+N_0) = x(n) \quad -\infty < n < \infty$$

Where N_0 is the Sampling period measured in units of number of Sample

* Periodic Signals can be represented in general

$$x_p(t) = \sum_{i=-\infty}^{\infty} x(t - iT_0)$$

Where $x(t) = \begin{cases} x(t), & t_1 \leq t < t_1 + T_0 \\ 0, & \text{else where.} \end{cases}$

$$x_p(n) = \left[\sum_{i=-\infty}^{\infty} x(n - iN_0), T\right]$$

Where $x(n) = \begin{cases} x(n), & n_1 \leq n < (n_1 + N_0) \\ 0, & \text{else where.} \end{cases}$

→ The Sum of two or more periodic continuous time Signals need not be periodic. They will be periodic if and only if the ratio of their fundamental periods is rational.

→ To determine whether the sum of two or more periodic signals is periodic or not, the steps are given below.

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Step-I Determine the fundamental period of the individual signals in the sum signal.

Step-II Find the ratio of the fundamental period of the first signal with the fundamental periods of every other signal.

Step-III If all these ratios are rational, then the sum signal is also periodic.

* In DTS, the sum of a no. of periodic signals is always periodic because the ratio of individual periods is always the ratio of integers, which is rational.

Ex - 1.1 (from Salibahan)

Even and odd Signals

→ If a signal exhibits symmetry in the time domain, it is called an even signal.

* The signal must be identical to its reflection about the origin.

ex - cosωt → For CTS → $x(t) = x(-t)$

For DTS → $x(n) = x(-n)$.

→ An odd signal exhibits anti-symmetry.

* The signal is not identical to its reflection about the origin, but to its negative.

ex - $x(t) = \sin\omega t$ → For CTS → $x(t) = -x(-t)$

For DTS → $x(n) = -x(-n)$.

A Signal can be expressed as a sum of two components, such as even component and odd component of the signal.

$$x(t) = x_{even}(t) + x_{odd}(t)$$

Where

$$x_{even}(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

$$x_{odd}(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

Energy & Power Signals

→ For DTS $x(n)$, the energy E is defined as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

→ The average power of a discrete time signal $x(n)$ is defined

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

Ex - 1.1 (from Ramesh Babu)

$$x(n) = \left(\frac{1}{3}\right)^n u(n)$$

$$\text{Energy } E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E = \sum_{n=0}^{\infty} \left[\left(\tfrac{1}{3}\right)^n \right]^2 \quad \therefore \; u(n) = 1 \text{ for } n \geq 0, \; = 0 \;\; n < 0$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n \quad \left[1 + a + a^2 + \cdots \infty = \frac{1}{1-a} \right]$$

$$= \frac{1}{1 - \frac{1}{9}} = 9/8$$

Power (Avg) $P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{9}\right)^n$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \left[\frac{1 - (1/9)^{N+1}}{1 - 1/9} \right]$$

$$= 0$$

* If Energy Signal 'E' is finite, the $P_{av} = 0$

→ If 'E' is infinite, then the Power may be either finite or infinite.

Classifications of System

Signals → discrete Signals, continuous Signal

→ Both continuous and discrete time signals are classified into
→ Static and dynamic Systems
→ Linear and Non-Linear Syst.
→ Time Variant and time invariant Syst
→ Causal and Non-causal Syst.
→ Stable and Unstable Syst.

* Static and dynamic Systems

→ A DTS is called Static or memoryless if its output at any instant 'n' depends on the input sample at the same time, but not on past or future samples of the input.

* In any other case, the system is said to be dynamic or to have memory.

Ex. Static/memoryless :

$$y(n) = ax(n)$$

$$y(n) = ax^2(n).$$

Dynamic/Memory :

$$y(n) = x(n-1) + x(n-2).$$

Causal and Non-causal System

→ A System is said to be causal if the output of the System at any time n depends only on present and past inputs, but doesn't depends on future inputs.

$$y(n) = f[(x(n), x(n-1), x(n-2) \ldots]$$

→ If the System depends not only on present and past inputs but also on future inputs, then it is said to be Non-causal.

Ex. 1.4 (Ramesh Babu)

Causal System → $y(n) = x(n) - x(n-1)$

$y(n) = ax(n)$.

Non-causal System → $y(n) = x(n) + 3x(n+4)$.

$y(n) = x(n^2)$

For $n=-1$ $y(-1) = x(1)$

For $n=0$ $y(0) = x(0)$

For $n=1$ $y(1) = x(1)$.

Linear and Non-Linear Systems.

- A System that satisfies the superposition principle, called Linear System.

→ A System is linear if and only if

$$T[a_1x_1(n)+a_2x_2(n)] = a_1T[x_1(n)]+a_2T[x_2(n)]$$

for any arbitrary constants a_1 and a_2

* A relaxed System that doesn't satisfy the superposition principle, called Non-linear System.

Ex- $y(n) = x^2(n)$

The output due to the signals $x_1(n)$ & $x_2(n)$ are

$$y_1(n) = T[x_1(n)] = x_1^2(n)$$

$$y_2(n) = T[x_2(n)] = x_2^2(n)$$

The Weighted Sum of output is

$$a_1T[x_1(n)]+a_2T[x_2(n)]$$

$$= a_1x_1^2(n)+a_2x_2^2(n) \quad \text{--- (1)}$$

* $y_3(n) = y_1(n)+y_2(n) = T[x_1(n)]+T[x_2(n)]$

output due to Weighted Sum of inputs is

$$y_3(n) = T[a_1x_1(n)+a_2x_2(n)] = [a_1x_1(n)+a_2x_2(n)]^2 \quad \text{--- (2)}$$

eqn (1) ≠ eqn (2)

Non-linear

Ex $y(n) = nx(n) \longrightarrow$ Linear System

* Time variant and Time-invariant System.

$\rightarrow$ A System is called time-invariant if its input-output characteristics donot change with time.

$\longrightarrow$ Delay the input sequence by 'K' samples & find output sequence,

$$y(n,K) = T[x(n-k)]$$

$\rightarrow$ Delay the output sequence by 'K' samples ie $y(n-k)$

$$\Rightarrow y(n,k) = y(n-k)$$

If, for possible values of 'K'

$$y(n,k) \neq y(n-k)$$

System is time-variant.

Ex - 1.3 (Ramesh Babu)

(i) $y(n) = x(n) + x(n-1)$

Soln:- $y(n) = T[x(n)] = x(n) + x(n-1)$.

$\rightarrow$ Input is delayed by 'K' units

$$y(n,k) = T[x(n-k)] = x(n-K) + x(n-K-1).$$

$\rightarrow$ output is delayed by 'K' units in time.

$$y(n-k) = x(n-K) + x(n-K-1)$$

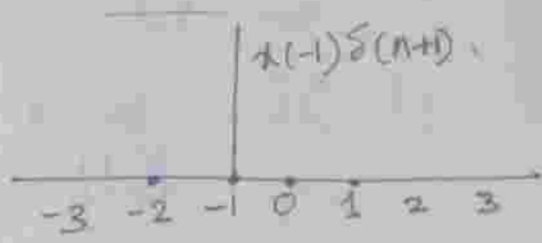
So, $y(n,k) = y(n-k)$. $\rightarrow$ It is time-invariant.

Representation of Arbitary Sequence.

$$\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

for $k=1$ $\quad \sum_{k=-\infty}^{\infty} x(1)\,\delta(n-1)$.

for $k=-1$ $\quad \sum_{k=-\infty}^{\infty} x(-1)\,\delta(n+1)$.



Ex-1.6 Represent the sequence $x(n)=\{4, 2, -1, \underset{\uparrow}{1}, 3, 2, 1, 5\}$ as a sum of shifted unit impulses.

Soln

$$x(n) = \{4, 2, -1, \underset{\uparrow}{1}, 3, 2, 1, 5\}$$

$\quad\quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

$n = -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

$$x(n) = x(-3)\delta(n+3) + x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + x(3)\delta(n-3) + x(4)\delta(n-4).$$

$$= 4\delta(n+3) + 2\delta(n+2) - \delta(n+1) + \delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3) + 5\delta(n-4).$$

Stable and Unstable

→ A System is said to BIBO Stable if every bounded input produces a bounded output.

→ A bounded Signal has an amplitude that remain finite.

* The conditions for a System to be BIBO Stable are —

(i) If the Syst. transfer function is a rational function, the degree of the numerator must not be larger than the degree of the denominator.

(ii) The poles of the Syst. must lie in the left half of the S-plane or within the unit circle in the Z-plane.

(iii) If a pole lies on the imaginary axis, it must be a single-order one, i.e no repeated poles must lie on the imaginary axis.

The Systems not satisfying the above conditions are unstable.

Mathematically

Let $x(n)$ be a bounded i/p sequence, $h(n)$ be the impulse response of the System and $y(n)$ be the output sequence.

→ Magnitude of the output

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \right|$$

We know that the magnitude of the sum of the terms is less than or equal to the sum of the magnitude, then $|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)|$

Consider the bounded value of the i/p is equal to 'M'.

$$\text{So, } |y(n)| \leq M \sum_{k=-\infty}^{\infty} h(k).$$

→ The above condition will be satisfied when

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty.$$

→ The necessary & sufficient condition for stability is

$$\boxed{\sum_{n=-\infty}^{\infty} |h(n)| < \infty}.$$

Ex. 1.15 check the stability -

$$h(n) = \left(\tfrac{1}{2}\right)^n u(n).$$

Soln Stability condition is $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n u(n) \qquad u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for} < 0 \end{cases}.$$

$$\text{or } \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n = 1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \cdots \infty$$

$$= \frac{1}{1-\frac{1}{2}} = \tfrac{1}{2} < \infty$$

→ System is stable.

$$\left[\because 1 + a + a^2 \ldots \infty = \frac{1}{1+a} \right]$$

* [FIR & IIR] Exams

Convolution Sum

$$x(n) \rightarrow \boxed{T} \rightarrow y(n) = T[x(n)]$$

→ unit impulse (i/p) $x(n) = \delta(n)$, then o/p of the system is known as impulse Response denoted $h(n)$.

$$h(n) = T[\delta(n)].$$

→ We know that any arbitrary sequence $x(n)$ can be represented as a weighted sum of discrete impulse. So,

$$y(n) = T[x(n)] = T\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] \quad \text{—— (1)}$$

For linear system it reduces the above eqn

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, T[\delta(n-k)] \quad \text{—— (2)}$$

* Response to the shifted impulse sequence can be denoted by $h(n,k)$ is defined

$$h(n,k) = T[\delta(n-k)] \quad \text{—— (3)}$$

We know that for a time invariant system

$$h(n,k) = h(n-k). \quad \text{—— (4)}$$

By comparing eqn (3) & (4) we can write

$$T[\delta(n-k)] = h(n-k) \longrightarrow (5)$$

By substituting eqn (5) in eqn (2) we can get that

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad \text{—— (6)}$$

→ For LTI System if the i/p sequence $x(n)$ and impulse response $h(n)$ is given, We can find $y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$.

↓

Convolution Sum

$y(n) = x(n) * h(n)$ $*$ → denotes convolution sum.

* Procedure to find out the convolution sum of two sequence :—

Step-I choose an initial value of 'n', the starting time for evaluating the output sequence $y(n)$. If $x(n)$ starts at $n = n_1$ and $h(n)$ starts at $n = n_2$ then $n = n_1 + n_2$ is a good choice.

Step-II Express both ~~sequence~~ seq in terms of the index 'k'.

Step-III Fold $h(k)$ about $k = 0$ to obtain $h(-k)$ and shift by 'n' to the right if n is +ve and left if 'n' is -ve to obtain $h(n-k)$.

Step-IV Multiply the two sequence $x(k)$ and $h(n-k)$ element by element and sum the products to get $y(n)$.

Step-V. Increment the index 'n', shift the sequence $h(n-k)$ to right by one sample & do Step-IV.

Step-VI - Repeat Step-V until the sum of products is zero for all remaining values of 'n'.

Properties of convolution

→ Commutative law: $x(n) * h(n) = h(n) * x(n)$.

→ Associative law:- $[x(n) * h_1(n)] * h_2(n)$
$= x(n) * [h_1(n) * h_2(n)]$

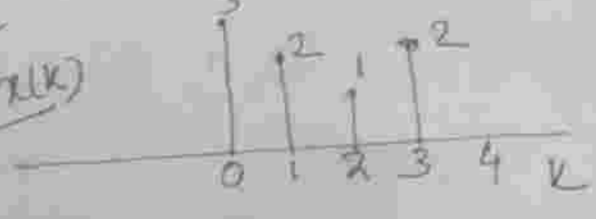
→ Distributive law:-
$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$.

Ex-1.7 (Ramesh Babu)

Determine the convolution sum of two sequences $x(n) = \{3, 2, 1, 2\}$ $h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$

Soln

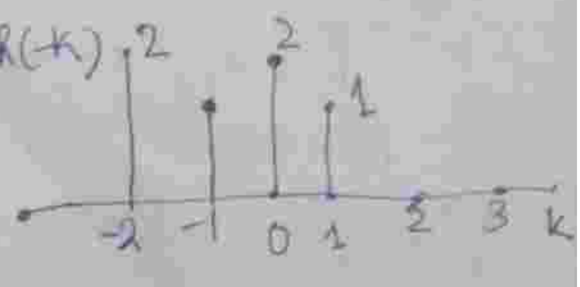
$x(n) = x(k)$



$h(n) = h(k)$

$h(-1-k)$

$h(-k)$



To check the correctness of the problem

$\sum x(n) = 8$, $\sum h(n) = 6$

$\sum y(n) = \sum x(n) \cdot \sum h(n) = 8 \times 6 = 48$

Method-2

k		-4	-3	-2	-1	0	1	2	3	4	5	6	7
x(k)						3	2	1	2				
n=-1	h(-1-k)		2	1	2	1							
n=0	h(0-k)			2	1	2	1						
n=1	h(1-k)				2	1	2	1					
n=2	h(2-k)												
n=3	h(3-k)												
n=4	h(4-k)												
n=5	h(5-k)												

$y(-1) = 3 \cdot 1 = 3$, $y(0) = 3 \cdot 2 + 2 \cdot 1 = 8$

$y(1) = 3 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 = 8$

Method-3

x(n)

	3	2	1	2
1	3	2	1	2
2	6	4	2	4
1	3	2	1	2
2	6	4	2	4

Results $\Rightarrow y(n) = \{3, 8, 8, 12, 9, 4, 4\}$

* Length of the Sequence $x(n) = N_1$ and $h(n) = N_2$

So, Length of the convolution Sum $y(n) = N_1 + N_2 - 1$

→ The limits in the convolution sum can be modified according to the type of sequence & system.

* For a causal system the impulse response $h(n) = 0$ for $n < 0$.

Convolution for causal system –

$$y(n) = \sum_{k=-\infty}^{n} x(k)\, h(n-k).$$

$$= \sum_{k=0}^{\infty} h(k)\, x(n-k).$$

Ex-1.9

Find $y(n)$ if $x(n) = n+2$ for $0 \leq n \leq 3$

$h(n) = a^n u(n)$ for all n.

Solⁿ

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Given $x(n) = n+2$ for $0 \leq n \leq 3$

$h(n) = a^n u(n)$ for all n.

$h(n) = 0$ for $n < 0$. So, the syst is causal.

$x(n)$ is a causal finite sequence whose value is zero for $n > 3$.

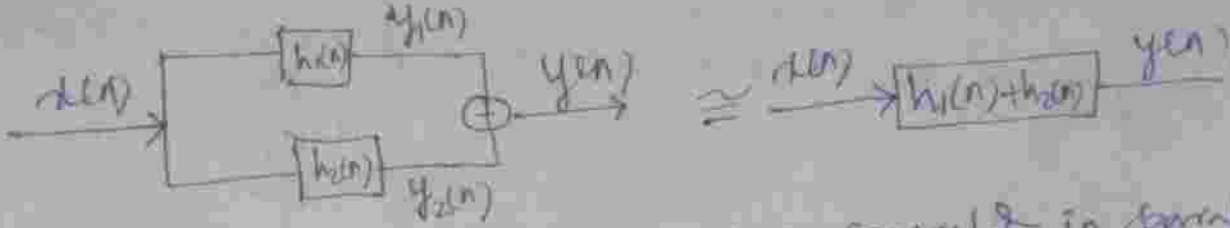
So, $$y(n) = \sum_{k=0}^{3} x(k)\, h(n-k).$$

$$= \sum_{k=0}^{3} (k+2)\, a^{n-k}\, u(n-k)$$

$$y(n) = 2a^n u(n) + 3a^{n-1} u(n-1) + 4a^{n-2} u(n-2) + 5a^{n-3} u(n-3).$$

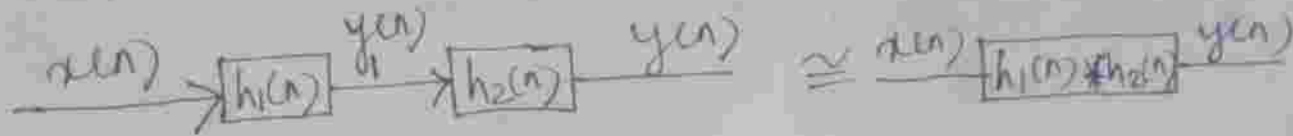
Interconnection of LTI System

(i) Parallel connection of systems.



* If the two systems are connected in parallel, the overall response is equal to the sum of two impulse responses.

(ii) Cascade connection of two systems →



* The impulse response of two LTI systems connected in cascade is the convolution of the individual impulse response.

Correlation of two Sequence

Correlation is basically used to compare two signals. It occupies a significant place in signal processing. It has application in radar & sonar system where the location of the target is measured by comparing the transmitted & reflected signals.

Other application is image processing & control engineering.

Def^n:- Correlation is a measure of the degree to which two signals are similar.

(i) Cross-correlation (ii) Auto-correlation

Cross - correlation :-

→ Cross - correlation between a pair of signals $x(n)$ and $y(n)$ is given

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l) \quad l = 0, \pm 1, \pm 2, \pm 3 \ldots \quad \text{——} (1)$$

(lag)

Where 'l' is the shift parameter.

order of subscripts 'xy' indicates that $x(n)$ is the reference sequence that remains unshifted in time where $y(n)$ is shifted 'l' units in time with respect to $x(n)$.

* If we want to fix $y(n)$ and to shift $x(n)$, then correlation can be written

$$\gamma_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-l)$$

$$= \sum_{n=-\infty}^{\infty} y(n+l)\, x(n). \quad \text{——} (2)$$

* If the time shift $l = 0$, then

$$\gamma_{xy}(0) = \gamma_{yx}(0) = \sum_{n=-\infty}^{\infty} x(n)\, y(n).$$

By comparing eqn (1) & (2),

$$\gamma_{xy}(l) = \gamma_{yx}(-l)$$

$\gamma_{xy}(-l)$ is the folded version of $\gamma_{xy}(l)$ about $l=0$.

We can rewrite (1)

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y[-(l-n)]$$

$$= x(l) * y(-l). \quad \text{——} (3)$$

Autocorrelation ($\gamma_{xx}(l)$)

→ The autocorrelation of a sequence is correlation of a sequence with itself.

Autocorrelation of a sequence $x(n)$ is defined

$$\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l).$$

$$= \sum_{n=-\infty}^{\infty} x(n+l)\, x(n)$$

→ If the time shift $l=0$, then

$$\gamma_{xx}(0) = \sum_{n=-\infty}^{\infty} x^2(n).$$

* Properties of cross-correlation & Auto-correlation Sequence :-

$$\rho_{xx}(l) = \frac{\gamma_{xy}(l)}{\gamma_{xx}(0)}$$

This is normalised expression for $\gamma_{xx}(l)$.

→ The normalised cross-correlation sequence is

$$\rho_{xy}(l) = \frac{\gamma_{xy}(l)}{\sqrt{\gamma_{xx}(0)\,\gamma_{yy}(0)}}$$

$\rho_{xy}(l)$ is also known as the cross-correlation coefficient. Its value always lies between -1 & +1.

→ A value '0' for cross-correlation means no correlation.

Ex 1.13 Find the cross correlation of two finite length sequence $x(n) = \{\underset{\uparrow}{1}, 2, 1, 1\}$; $y(n) = \{\underset{\uparrow}{1}, 1, 2, 1\}$

Sol^n $x(l) = \{\underset{\uparrow}{1}, 2, 1, 1\}$ $y(-l) = \{\underset{\uparrow}{1}, 2, 1, 1\}$

$$r_{xy}(l) = x(l) * y(-l).$$

		$y(-l)$		
	1	2	1	1
1	1	2	1	1
$x(l)$ 2	2	4	2	2
1	1	2	1	1
1	1	2	1	1

$$r_{xy}(l) = \{\underset{\uparrow}{1}, 4, 6, 6, 5, 2, 1\}$$

computation of correlation

$$r_{xy}(l) = x(l) * y(-l).$$

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correlation of Power and periodic Signals

Consider $x(n)$ & $y(n)$ are the power signals. Their cross-correlation sequence is defined

$$\gamma_{xy}(l) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, y(n-l)$$

→ Auto-correlation of Power Signals

$$\gamma_{xx}(l) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, x(n-l).$$

* If $x(n)$ & $y(n)$ are periodic sequences, each with period N, then

$$\gamma_{xy}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, y(n-l)$$

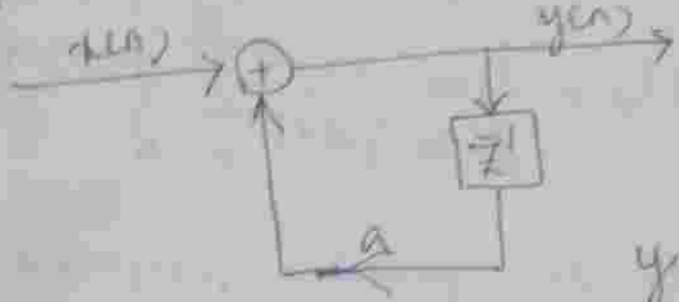
&

$$\gamma_{xx}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, x(n-l).$$

Linear time-Invariant Syst. characterized by constant co-efficient Difference Equⁿs —

→ A family of LTI systems described by an input-output relation called a difference equation with constant co-efficients.

→ Systems described by constant co-efficient is linear difference equation. (Recursive Syst)



$y(n) = ay(n-1) + x(n)$

↓ independent of time.

Compute successive value of $y(n)$ for $n \geq 0$.

for $n=0$, $y(0) = ay(-1) + x(0)$

$n=1$, $y(1) = ay(0) + x(1) = a\{ay(-1) + x(0)\} + x(1)$

$= a^2y(-1) + ax(0) + x(1)$.

$n=2$, $y(2) = ay(1) + x(2)$

$= a\{a^2y(-1) + ax(0) + x(1)\} + x(2)$

$= a^3y(-1) + a^2x(0) + ax(1) + x(2)$

⋮

$y(n) = ay(n-1) + x(n)$

$= a^{n+1}y(-1) + a^n x(0) + a^{n-1}x(1) + \dots$

$+ ax(n-1) + x(n)$.

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* More Accurately / compactly

$$y(n) = \underbrace{a^{n+1} y(-1)}_{\text{initial condition } y(-1)} + \underbrace{\sum_{k=0}^{n} a^k x(n-k)}_{\text{input Signal } x(n)}, \quad n \geq 0$$

→ A Recursive System is relaxed if it starts with zero initial condition.

→ Initially relaxed at time $n=0$ means memory should be zero. Memory describes its state i.e zero state & its corresponding output is called zero-state response or forced response. It is denoted by $y_{zs}(n)$.

$$y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k), \quad n \geq 0.$$

→ The output of the system with zero input is called the zero-input response or natural response $y_{zi}(n)$. (free Response)

$$y_{zi}(n) = a^{n+1} y(-1).$$

$$y(n) = y_{zi}(n) + y_{zs}(n).$$

* The general form of linear constant co-efficient difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

or $$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) . \quad (\because a_0 = 1)$$

...ere 'N' is called the $\frac{\text{continuing}}{\text{order}}$ of the $\frac{\text{Page - 16}}{\text{difference}}$ equations.

* A System is linear if it satisfies

→ $y(n) = y_{zi}(n) + y_{zs}(n)$.

→ The principle of superposition applies to the zero-state response. (zero state linear)

→ The principle of superposition applies to the zero-input response (zero-input linear).

*→ for a relaxed syst ⇒ $y_{zi}(n) = 0$.

Solution Of Linear constant-co-efficient Difference equations

(i) Direct method
(ii) Indirect method (Z-transform approach)

→ The direct solution method says that the total solution is the sum of two parts.

$$y(n) = y_h(n) + y_p(n).$$

$y_h(n) \rightarrow$ homogeneous / complementary solⁿ
$y_p(n) \rightarrow$ Particular Solution.

* Homogeneous Solution of a difference equation

Assuming $x(n) = 0$,
Solⁿ to the homogeneous difference equation

$$\sum_{k=0}^{N} a_k y(n-k) = 0$$

Solⁿ → Let $y_h(n) = \lambda^n$

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0$$

$$\Rightarrow \lambda^{n-N} (\underbrace{\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_{N-1}\lambda + a_N}_{\text{characteristic equation of the syst}}) = 0$$

It has 'N' roots. Roots are $\lambda_1, \lambda_2 \ldots \lambda_N$.
The roots are real or complex valued.
$a_1, a_2, a_3 \ldots a_N$ are usually real.

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The most general solution Page-17
to the homogeneous difference equation
is $y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \cdots + C_N\lambda_N^n$

Where $C_1, C_2, \ldots C_N$ are weighting co-efficients.

→ These coefficients are determined from the initial conditions specified for the system.
input $x(n)=0$ can be used to obtain the zero-input response of the system.

Ex - Determine the homogeneous soln of the system described by the first order difference equation

$$y(n) + a_1 y(n-1) = x(n).$$

Soln → By setting $x(n)=0$.

$$y(n) + a_1 y(n-1) = 0$$

Let $y(n) = \lambda^n$

$\Rightarrow \lambda^n + a_1 \lambda^{n-1} = 0$

$\Rightarrow \lambda^{n-1}(\lambda + a_1) = 0$

$\lambda = -a_1$

$n=0,$

$y(0) = -a_1 y(-1).$

$y_h(0) = c$ $c = -a_1 y(-1)$

$y_h(n) = c\lambda^n = c(-a_1)^n$

$= -a_1 y(-1)(-a_1)^n$

$= (-a_1)^{n+1} y(-1)$

→ The soln to the homogeneous difference equation is $y_h(n) = c\lambda^n = c(-a_1)^n$

→ For first order diffrence eqn $y_{zi}(n) = a^{n+1} y(-1).$

$$y_{zi}(n) = (-a_1)^{n+1} y(-1).$$

102 Problems

Ex-2 Determine the zero-input response of the system described by the homogeneous second order difference equation

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

Soln

Let $y_h(n) = \lambda^n$

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$

$$\Rightarrow \lambda^{n-2} \underbrace{(\lambda^2 - 3\lambda - 4)}_{} = 0 \longrightarrow \text{characteristic equation}$$

$$\boxed{\lambda_1 = -1 \ \& \ \lambda_2 = 4}$$

homogeneous Soln $\quad y_h(n) = c_1\lambda_1^n + c_2\lambda_2^n$

$$= c_1(-1)^n + c_2(4)^n$$

→ Zero-input response can be calculated from homogeneous soln by evaluating the constants (c_1, c_2). given initial condition $y(-1)$ & $y(-2)$.

for $n=0$, $y(0) = c_1 + c_2$

$n=1 \quad y(1) = -c_1 + 4c_2$

$$y(n) = 3y(n-1) + 4y(n-2)$$

$$y(0) = 3y(-1) + 4y(-2)$$

$$y(1) = 3y(0) + 4y(-1)$$

$$= 3\{3y(-1) + 4y(-2)\} + 4y(-1).$$

We can write $= 13y(-1) + 12y(-2)$.

$$c_1 + c_2 = 3y(-1) + 4y(-2)$$

$$-c_1 + 4c_2 = 13y(-1) + 12y(-2).$$

After solving the above two equations,

we ~~can~~ get, $c_1 = -\frac{1}{5} y(-1) + \frac{4}{5} y(-2)$,

$c_2 = \frac{16}{5} y(-1) + \frac{16}{5} y(-2)$.

→ Zero-input response of the system is

$$\therefore y_{zi}(n) = \left[-\frac{1}{5} y(-1) + \frac{4}{5} y(-2)\right](-1)^n + \left[\frac{16}{5} y(-1) + \frac{16}{5} y(-2)\right](4)^n$$

$\boxed{\text{* for example } y(-2) = 0 \;\&\; y(-1) = 5, \text{ then } c_1 = -1 \;\&\; c_2 = 16, \; y_{zi}(n) = (-1)^{n+1} + (4)^{n+2}}$

* If the characteristic equation contains multiple roots, then the solution in the form of

$$y_h(n) = c_1\lambda_1^n + c_2 n\lambda_1^n + c_3 n^2\lambda_1^n + \cdots + c_m n^{m-1}\lambda_1^n + c_{m+1}\lambda_{m+1}^n + \cdots + c_N\lambda_N^n.$$

The particular solution of the difference equation
↓ $y(P)$

It depends on specified input signal $x(n)$.

$y_p(n)$ is any solution satisfying

$$\sum_{k=0}^{N} a_k y_p(n-k) = \sum_{k=0}^{M} b_k x(n-k) \quad \underline{a_0 = 1}$$

General form of the particular Solⁿ for several types of input signals

Input Signal $x(n)$	Particular Solution $y_p(n)$
A (constant)	K
AM^n	KM^n
An^M	$K_0 n^M + K_1 n^{M-1} + \cdots + K_M$
$A^n n^M$	$A^n (K_0 n^M + K_1 n^{M-1} + \cdots + K_M)$
$\left\{\begin{matrix} A\cos\omega_0 n \\ A\sin\omega_0 n \end{matrix}\right\}$	$K_1 \cos\omega_0 n + K_2 \sin\omega_0 n$

Ex-2.4.7 Determine the particular Solⁿ of the difference equation $y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n)$ when the forcing function $x(n) = 2^n,\ n \geq 0$; 0 elsewhere.

Solⁿ

$$y_p(n) = K2^n \quad n \geq 0$$

$$K2^n u(n) = \frac{5}{6}K2^{n-1}u(n-1) - \frac{1}{6}K2^{n-2}u(n-2) + 2^n u(n)$$

To evaluate the value of K, for $n \geq 2$.

$$4K = \frac{5}{6}(2K) - \frac{1}{6}K + 4$$

$$K = 8/5.$$

→ The particular Solution is $y_p(n) = \frac{8}{5}2^n,\ n \geq 0.$

Z-transform

→ The Z-transform is used to ~~f~~ transform of DT Signals.

→ It is similar to that of Laplace transform for analog Signals.

* The poles and zeros of transfer function are used for stability analysis of the DT systems.

* The basic difference between the two transforms is that the s-plane is arranged in a rectangular co-ordinate system, while the Z-transform uses a polar format.

Need of a Transform

A transform helps in three ways

→ The Spectral properties of the Signal are better revealed in a transform domain.

→ The System transfer function in a transform domain directly indicates the locations of poles and zeros which helps in analysing Stability of the system.

→ A Signal, while transmission over a communication channel, needs to be compressed.

** Relation between Laplace transform and Z-transform

$$\boxed{z = e^{sT}}$$

** Relation between Fourier Transform (FT) and Z-transform.

$$Z = r e^{j\Omega}$$

$r \rightarrow$ represent the magnitude of Z

$\Omega \rightarrow$ represent the angle or phase of Z.

$$F(re^{j\Omega}) = \sum_{n=0}^{\infty} f(nT)(re^{j\Omega})^{-n}$$

As $F(Z) = \sum_{n=0}^{\infty} f(nT) Z^{-n}$

$$F(Z) = \sum_{n=0}^{\infty} [f(n) r^{-n}] e^{-j\Omega n}$$

→ Analog Signal may exist from $-\infty$ to $+\infty$.
So, we write

$$F(Z) = \sum_{-\infty}^{\infty} [f(n) r^{-n}] e^{-j\Omega n}.$$

for $r=1$ or equivalently for $Z=1$, the Z-transform reduces to FT.

→ Z-transform reduces to FT on the unit circle in complex Z-plane.

* Convergence of Z-transform (ZT) requires the convergence of sequence $f(n) r^{-n}$.

→ The range of values of r for which the sequence $f(n) r^{-n}$ converges is called the Region of convergence (ROC).

** If ROC includes the unit circle then the FT also converges.

Analysis of LTI ~~System~~ continuous time system using Laplace transform. The counterpart of Laplace transform is called Z-transform for LTI Discrete time Systems. ③

→ Laplace transform converts differential equation to algebraic equation.

→ Z - transform converts difference equation to algebraic equation.

* The Z-transform comes into two varieties.
  - → Bilateral or two sided ZT
  - → Unilateral or one - sided ZT

⇒ Bilateral Z-transform offers insight into the nature of System characteristics such as Stability, causality and frequency response.

⇒ Uniteral Z-transform is convenient tool for solving difference equation with initial conditions.

given Signal x(n)

$$Z[x(n)] = X(Z) = \sum_{-\infty}^{\infty} x(n) Z^{-n}$$

⇒ Inverse Z-transform of X(Z) is

$$x(n) = \frac{1}{2\pi j} \oint_c X(Z) Z^{n-1} dz$$

** Transform relationship between $x(n)$ & $X(z)$
as $x(n) \longleftrightarrow X(z)$

→ Relationship between Z-transform and Discrete-Time Fourier transform

Discrete time signal $x(n)$. Its Z-transform is as $Z[x(n)] = X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$

Let $z = re^{j\omega}$

$$Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] e^{-j\omega n}$$

$$= F[x(n) r^{-n}]$$

$$Z[x(n)] = F[x(n) r^{-n}]$$

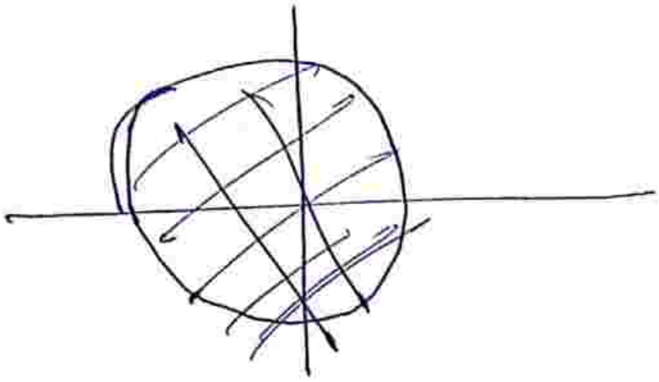
Thus, the Z-transform of $x(n)$ is the DTFT of $x(n) r^{-n}$.

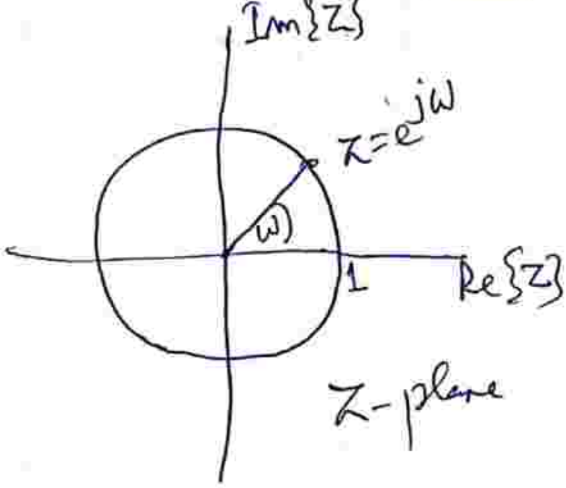
Now if $r = 1$, we obtain

$$Z[x(n)] = F[x(n)] \text{ for } r = 1$$

The Z-transform reduces to the DTFT when $|z| = 1$ (i.e. $z = e^{j\omega}$)

→ The z-transform reduces to the DTFT on the contour in the complex z-plane corresponding to a circle with radius unity.





Poles and Zeros

Right-hand Sequence

→ A RH sequence is one for which $x(n)=0$ for all $n<n_0$ where n_0 is +ve or -ve but finite.

→ If n_0 is greater than equal to zero, the resulting sequence is causal or a +ve time sequence.

→ For such type of sequence, the ROC is entire z-plane except $z=0$

Ex $x(n)=\{\underset{\uparrow}{1},0,3,-1,2\}$

$$X(Z) = x(0)+x(1)Z^{-1}+x(2)Z^{-2}+\cdots$$

$x(0)=1$ $x(1)=0$ $x(2)=3$ $x(3)=-1$ $x(4)=2$

$$X(Z)=1+3z^{-2}-z^{-3}+2z^{-4}.$$

Left hand Sequence

→ A left hand sequence $x(n)$ is one for which $x(n) = 0$ for all $n \geq n_0$, where n_0 is +ve or -ve but finite. If $n_0 \leq 0$, the sequence is anticausal sequence.

→ For such type of sequence, the ROC is entire z-plane except at $z = \infty$

Ex $x(n) = \{-3, -2, -1, 0, \underset{\uparrow}{1}\}$

$x(0) = 1$, $x(-1) = 0$ $x(-2) = -1$ $x(-3) = -2$
$x(-4) = -3$

$X(z) = 1 - z^2 - 2z^3 - 3z^4$

Two Sided Sequence

Find the z-transform of the sequence

$x(n) = \{2, -1, 3, 2, \underset{\uparrow}{1}, 0, 2, 3, -1\}$

$X(z) = ?$

Initial value theorem

$$x(0) = \lim_{z \to \infty} X(z)$$

Final value theorem

$$x(\infty) = \lim_{z \to 1} (1-z^{-1}) X(z), \text{ if } (1-z^{-1})X(z)$$

have no pole on or outside the unit circle.

** Parseval theorem

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1}\, dv$$

Ex Find the z-transform of $x(n) = n a^n u(n)$

we know that-

Sol^n $Z[a^n u(n)] = \dfrac{1}{1-az^{-1}}$ $|z| > a$

By differentiation property,

$$Z[n\, x(n)] = -z \frac{d}{dz} X(z)$$

$$Z[n a^n u(n)] = -z \frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right]$$

$$= \frac{az^{-1}}{(1-az^{-1})^2} \quad |z| > |a|$$

Sequence	z-transform	ROC
1. $\delta(n)$	1	All z
2. $u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u(-n-1)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta(n-m)$	z^{-m}	All z except 0 (if $m>0$); All z except ∞ (if $m<0$)
5. $a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $n a^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-n a^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$.

z-transform properties

Linearity $\rightarrow a_1x_1(n)+a_2x_2(n) \rightarrow a_1X_1(z)+a_2X_2(z)$

* Time Shifting $x(n-m) \rightarrow z^{-m}X(z)$

* Time Reversal $x(-n) \longrightarrow X(z^{-1})$

* Differentiation in z domain

$$nx(n) \longrightarrow -z\frac{d}{dz}X(z)$$

"Z-transform"

* The Direct Z-transform

→ The Z-transform of a discrete-time signal $x(n)$ is defined as power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

Where z - is the complex variable.

→ It is called the direct Z-transform because it transforms the time domain signal $x(n)$ into its complex plane representation $X(z)$.

→ The Z-transform of a signal $x(n)$ is denoted by

$$X(z) = Z\{x(n)\}$$

Relation between $x(n)$ & $X(z)$ is indicated by

$$x(n) \xleftrightarrow{z} X(z)$$

→ Z - transform is an infinite power series, it exists only for those values of Z. for which this series convergence.

* The Region of convergence (ROC) of $X(z)$ is the set of all values of Z for which $X(z)$ attains a finite value.

Ex 3.1.1 Determine the Z-transform of the following finite duration signals.

(from Proakis, Page-152)

ex-2 Determine the z-transform of the signal
$$x(n) = \left(\tfrac{1}{2}\right)^n u(n).$$

Solⁿ ÷ The signal $x(n)$ consists of an infinite no. of non-zero values

$$x(n) = \left\{1, \tfrac{1}{2}, \left(\tfrac{1}{2}\right)^2, \left(\tfrac{1}{2}\right)^3 \ldots \left(\tfrac{1}{2}\right)^n \ldots\right\}$$

→ The z-transform of $x(n)$ is the infinite power series

$$X(z) = 1 + \tfrac{1}{2} z^{-1} + \tfrac{1}{2} z^{-2} + \ldots + \left(\tfrac{1}{2}\right)^n z^{-n} + \ldots$$

$$X(z) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\tfrac{1}{2} z^{-1}\right)^n$$

* Infinite geometric Series —

$$1 + A + A^2 + A^3 \ldots\ldots = \tfrac{1}{1-A} \text{ if } |A| < 1$$

So, $X(z) = \dfrac{1}{1 - \frac{1}{2} z^{-1}}$, ROC: $|z| > \tfrac{1}{2}$

* $\dfrac{\text{Zero}}{\text{Pole}} = \dfrac{N^r}{D^r}$ $\quad D^r = 0$

$1 - \tfrac{1}{2} z^{-1} = 0$

$\tfrac{1}{2} z^{-1} = 1 \Rightarrow z = \tfrac{1}{2}$

Region of convergence.

* Properties of the Z-transform

→ Linearity:

If $x_1(n) \xleftrightarrow{Z} X_1(z)$

& $x_2(n) \xleftrightarrow{Z} X_2(z)$

then $x(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{Z} X(z)$

$= a_1 X_1(z) + a_2 X_2(z)$.

a_1 & a_2 are constant.

→ Time Shifting:

If $x(n) \xleftrightarrow{Z} X(z)$

then $x(n-k) \xleftrightarrow{Z} z^{-k} X(z)$

* The ROC of $z^{-k}X(z)$ is the same as that of $X(z)$ except for $z=0$ if $k>0$ and $z=\infty$ if $k<\infty$.

→ Scaling in the Z-transform:

If $x(n) \xleftrightarrow{Z} X(z)$, ROC: $r_1<|z|<r_2$

then $a^n x(n) \xleftrightarrow{Z} X(a^{-1}z)$, ROC: $|a|r_1<|z|<|a|r_2$

~~where~~ for any constant 'a', real or complex.

→ Time reversal:-

If $x(n) \xleftrightarrow{Z} X(z)$, ROC: $r_1<|z|<r_2$

then $x(-n) \xleftrightarrow{Z} X(z^{-1})$ ROC: $\frac{1}{r_2}<|z|<\frac{1}{r_1}$

→ Differentiation in the z-domain –

If $x(n) \xleftrightarrow{z} X(z)$

then $n\,x(n) \xleftrightarrow{z} -z \dfrac{dX(z)}{dz}$

→ Convolution of two sequences –

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

Then $x(n) = x_1(n) * x_2(n) \xleftrightarrow{z} X(z) = X_1(z)\,X_2(z)$.

Ex 3.2.9 find the convolution $x(n)$ of the signals

$$x_1(n) = \{1, -2, 1\}$$

$$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

Sol^n

$$X_1(z) = 1 - 2z^{-1} + 1z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$X(z) = X_1(z)\,X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

(how it comes?)

$$x(n) = \{\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1\}$$

Aditi

Properties of ROC

(1) The ROC is a ring or disk in the z-plane centered at the origin.
(2) The ROC cannot contain any poles.
(3) If $x(n)$ is a causal sequence then the ROC is the entire z-plane except at $z=0$
(4) If $x(n)$ is a non-causal sequence, then the ROC is the entire z-plane except at $z=\infty$.
(5) If $x(n)$ is a finite duration, two sided sequence the ROC is entire z-plane except at $z=0$ & $z=\infty$.
(6) If $x(n)$ is an finite duration, two-sided sequence the ROC will consist of a ring in the z-plane, bounded on the interior & exterior by a pole, not containing any pole.
(7) The ROC of a LTI stable system contains the unit circle.
(8) The ROC must be a connected region.

* Initial value theorem

If $X_+(z) = Z\{x(n)\}$, then $x(0) = \lim_{z\to\infty} X_+(z)$

Proof:- $X_+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

As $z\to\infty$, all the terms vanish except $x(0)$, which proves the theorem.

ie $\lim_{z\to\infty} X_+(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x(n) z^{-n} = x(0)$.

2nd

Final value theorem

If $X_+(z) = Z\{x(n)\}$, where the ROC for $X_+(z)$ includes, but is not necessarily confined to, $|z|>1$ and $(z-1)X_+(z)$ has no poles on or outside the unit circle, then $\boxed{x(\infty) = \lim_{z \to 1} (z-1)X_+(z)}$.

Proof

correlation

If $X_1(z) = Z\{x_1(n)\}$ and $X_2(z) = Z\{x_2(n)\}$

then $Z[\gamma_{x_1x_2}(l)] = Z\left[\sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l)\right]$

$$= R_{x_1x_2}(z) = X_1(z)\, X_2(z^{-1})$$

* Z - transform of some common z-transfr.

→ Find the z-transform of the signal $x(n) = [3(3)^n - 4(2)^n]u(n)$

Sol^n :- $X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$

$$= \sum_{n=-\infty}^{\infty} [3(3)^n - 4(2)^n] u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} [3(3)^n - 4(2)^n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} 3(3)^n z^{-n} - \sum_{n=0}^{\infty} 4(2)^n z^{-n} = 3\sum_{n=0}^{\infty}(3z^{-1})^n - 4\sum_{n=0}^{\infty}(2z^{-1})^n$$

* First power series converges, when $|3z^{-1}| < 1$ ie $|z| > 3$.

* 2nd power series " , when $|2z^{-1}| < 1$, ie $|z| > 2$.

→ Take $|z| > 3$ as ROC

* The System function

→ The general form of constant co-efficient difference equation is

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Taking Z-transform on both sides, applying time shifting property, we obtain

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\Rightarrow Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k} \Big/ 1 + \sum_{k=1}^{N} a_k z^{-k}$$

→ transfer function / System function

→ If input $x(n) = \delta(n)$, $X(z)$ is equal to 1. then $Y(z) = H(z)$.

* Find the System function & impulse response of the System described by the difference eqn $y(n) = 1/5\, y(n-1) + x(n)$.

Sol^n $Y(z) = \frac{1}{5} z^{-1} Y(z) + X(z)$.

Inverse Z-transform

→ four methods and used for evaluating Z-inverse

* Long division
* Partial fraction expansion method
* Residue method.
* convolution method.

Long division

→ $$X(Z) = \frac{N(Z)}{D(Z)} = \frac{b_0 + b_1 Z^{-1} + \dots b_M Z^{-M}}{1 + a_1 Z^{-1} + \dots a_N Z^{-N}}$$

$$= x(0) + x(1) Z^{-1} + x(2) Z^{-2} + \dots$$

ex - $$X(Z) = \frac{Z + 0.2}{(Z + 0.5)(Z - 1)}, \quad Z > 1 \longrightarrow \text{ROC given}$$

Soln

$$\begin{array}{r|l} & Z^{-1} + 0.7 Z^{-2} + 0.8 Z^{-3} + 0.775 Z^{-4} \\ \hline Z^2 - 0.5Z - 0.5 & Z + 0.2 \\ & Z - 0.5 - 0.5 Z^{-1} \\ \hline & \quad 0.7 + 0.5 Z^{-1} \\ & \quad \vdots \end{array}$$

Partial fraction Method

③ Residue Method

If z - transform of a Sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

→ Multiplying both sides by z^{k-1} & integrating w.r.t z about a closed contour C in the ROC of $X(z)$, then

$$\oint_C X(z) z^{k-1} dz = \oint_C \sum_{n=-\infty}^{\infty} x(n) z^{-n+k-1} dz$$

$$= \sum_{n=-\infty}^{\infty} x(n) \oint_C z^{k-n-1} dz$$

Consider that 'C' encloses the origin of the z - plane. Then by Cauchy residue theorem

$$\boxed{\oint_C z^{k-n-1} dz = 2\pi j \delta_{kn}}$$

where $\delta_{kn} = \begin{cases} 1, & \text{if } k=n \\ 0 & \text{if } k \neq n \end{cases}$

Hence $\oint_C X(z) z^{k-1} dz = 2\pi j \sum_{n=-\infty}^{\infty} x(n) \delta_{kn}$

$$= 2\pi j x(k).$$

$$\boxed{\Rightarrow x(k) = \oint_C \frac{1}{2\pi j} X(z) z^{k-1} dz}$$

→ Therefore, the inverse z-transform relation is given by the contour integral

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

Where 'C' is a circle in the z-plane in the ROC of X(z).

→ The above equation can be evaluated by finding sum of all residues of the poles that is inside of the circle 'C'.

So, $x(n) = \sum$ [Residue of $X(z) z^{n-1}$ at the poles inside 'C']

$$\Rightarrow x(n) = \sum_i (z - z_i)\, X(z)\, z^{n-1} \Big|_{z = z_i}.$$

Ex 7.25 $X(z) = \dfrac{z+1}{\cancel{z}(z+0.2)(z-1)}$, $|z| > 1$

find inverse z by residue method.

Soln We know that $x(n) = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$

$= \sum$ residues of $X(z) z^{n-1}$ at poles of $X(z) z^{n-1}$ within 'C'.

$= \sum$ residues of $\dfrac{(z+1)\; z^{n-1}}{\cancel{z}(z+0.2)(z-1)}$ at poles of same within 'C'.

* ROC $|z| > 1$ encloses the poles at $z = -0.2$ & $z = 1$ and for $n = 0$ the pole is at $z = 0$.

$\rightarrow$ So, for $n=0$

$$x(0) = \sum \text{residues of } \frac{z+1}{z(z+0.2)(z-1)} \text{ at poles } z=0,\ 1\ \&\ -0.2$$

$$= \cancel{z}\, \frac{z+1}{\cancel{z}(z+0.2)(z-1)}\Bigg|_{z=0} + \frac{(z+0.2)\,(z+1)}{z(z+0.2)(z-1)}\Bigg|_{z=-0.2}$$

$$+ (z-1)\,\frac{(z+1)}{z(z+0.2)(z-1)}\Bigg|_{z=1}$$

$$\Rightarrow x(0) = -5 + 10/3 + 5/3 = 0$$

$$\boxed{\Rightarrow x(0) = 0}$$

for $\underline{\underline{n \geq 1}}$

$$x(n) = \sum \text{residues of } \frac{(z+1)\,z^{n-1}}{(z+0.2)(z-1)} \text{ at poles } z=-0.2\ \&\ z=1$$

$$= \text{Residue of } \frac{(z+1)\,z^{n-1}}{(z+0.2)(z-1)}\Bigg|_{z=-0.2}$$

$$+ \text{residue of } \frac{(z+1)\,z^{n-1}}{(z+0.2)(z-1)}\Bigg|_{z=1}$$

$$= \frac{(z+1)\,z^{n-1}}{(z-1)}\Bigg|_{\text{at } z=-0.2} + \frac{(z+1)\,z^{n-1}}{(z+0.2)}\Bigg|_{z=1}$$

$$= -2/3\,(-0.2)^{n-1} + 5/3$$

$$\boxed{\Rightarrow x(n) = -2/3\,(-0.2)^{n-1}\,u(n-1) + 5/3\,u(n-1)}$$

2.26 Find Z-inverse by using Residue method

Page-7

$$X(Z) = \frac{Z}{(Z-2)(Z-3)}, \quad |Z|<2$$

Soln Two poles $Z=+3$ & $Z=2$ outside the ROC $|Z|<2$, So the sequence is Non-causal.

For $n<0$

$x(n) = -\sum$ Residues of $X(Z)\, Z^{n-1}$ at poles $Z=2$ & $Z=3$.

$$= -\left[\left.\frac{Z \cdot Z^{n-1}}{Z-3}\right|_{Z=2} + \left.\frac{Z \cdot Z^{n-1}}{(Z-2)}\right|_{Z=3} \right]$$

$$= -\left[-(2)^n + (3)^n\right] = (2)^n - (3)^n$$

for $n<0$ $x(n)$ can be written as

$$x(n) = \left[2^n - 3^n\right] u(-n-1).$$

Solution of difference equations using one sided Z-transform

By using time shift property

e 2.34 Use the one sided Z-transform to determine $y(n)$, $n \geq 0$ if $y(n) = \frac{1}{2} y(n-1) + x(n)$

$x(n) = \left(\frac{1}{3}\right)^n u(n)$; $y(-1) = 1$.

Soln By taking Z-transform of $y(n)$

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + Y(-1) + X(z)$$

from given data $Y(-1) = 1$.

* find the Step Response of the System given by $y(n) = a y(n-1) + x(n)$. Where $x(n) = u(n)$ & $y(-1) = 0$.

Soln Taking Z-transform of both sides

$$Y(z) = a z^{-1} Y(z) + y(-1) + X(z).$$

$$= a z^{-1} \left[Y(z) + y(-1) z \right] + X(z).$$

As $y(-1) = 0$

$$\Rightarrow Y(z) \left[1 - a z^{-1} \right] = X(z)$$

$$\Rightarrow Y(z) = \frac{1}{1 - a z^{-1}} \cdot X(z) = \frac{1}{1 - a z^{-1}} \cdot \frac{1}{1 - z^{-1}}$$

By using Partial fraction

$$Y(z) = \frac{1}{1-a} \left[\frac{1}{1 - z^{-1}} - \frac{a}{1 - a z^{-1}} \right]$$

On taking inverse Z-transform, we get

$$y(n) = \frac{1}{1-a}\left[u(n) - a(a)^n u(n)\right]$$

$$\boxed{y(n) = \frac{1}{1-a}\left[1 - a^{n+1}\right]u(n).}$$

Q. Determine the one-sided Z-transform of,

$$y(n) + \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) = 0, \quad y(-1) = y(-2) = 1$$

Sol^n: Taking Z-transform of difference equation

$$Y(z) + \frac{1}{2}\left[y(z)z^{-1} + y(-1)\right] - \frac{1}{4}\left[y(z)z^{-2} \overset{+y(-1)z}{+} y(-2)\right] = 0$$

$$y(z) + \frac{1}{2}z^{-1}\left[y(z) + y(-1)z\right] - \frac{1}{4}z^{-2}\left[y(z) + y(-2)z^{2}\right] = 0$$

given $y(-1) = y(-2) = 1$

~~$y(z) + \frac{1}{2}z^{-1}[y(z)+z$~~

$$y(z) + \frac{1}{2}\left[z^{-1}y(z) + y(-1)\right] - \frac{1}{4}\left[z^{-2}y(z) + y(-1)z^{-1} + y(-2)\right] = 0$$

$$y(z)\left[1 + \frac{z^{-1}}{2} - \frac{z^{-2}}{4}\right] = \frac{z^{-1}}{4} + \frac{1}{4} - \frac{1}{2}$$

$$Y(z) = \left(\frac{\frac{1}{4}z^{-1} - \frac{1}{4}}{1 + \frac{z^{-1}}{2} - \frac{z^{-2}}{4}}\right)$$

from e. Ramesh Babu
Dsni, page - 194

DFT

11/2/2016 Day-1

→ DFT is a powerful computation tool which allows to evaluate the FT (fourier transform) $X(e^{jw})$ on digital computer or specially hardware design.

→ DFT is defined only for sequence of finite length. $X(e^{jw})$ is continuous and periodic.

→ DFT is obtained by sampling one period of the FT at a finite number of frequency points.

* DFT plays an important role in the implementation of many signal processing algorithms.
* DFT is used to perform linear filtering operations in the frequency domain.

DFT of a sequence of periodic and in the frequency range 0 to 2π.

Many w in the range of 0 to 2π.

→ Use digital computer to compute 'N' equally spaced points over the interval of $0 \leq w \leq 2\pi$,

Then 'N' points should be located at

$$w_k = \frac{2\pi}{N} k, \quad k = 0, 1, 2 \ldots N-1.$$

⇒ 'N' equally spaced frequency samples of the DTFT are known as DFT. It denoted $X(k)$

$$X(k) = X(e^{jw})\Big|_{w = \frac{2\pi}{N}k}, \quad 0 \leq k \leq N-1.$$

Pag

$$\underset{X(K)}{\text{DFT}} = \sum_{k=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}, \quad 0 \le k \le N-1$$

$$\underset{x(n)}{\text{IDFT}} = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}, \quad 0 \le n \le N-1$$

* $W_N = e^{-j2\pi/N} \rightarrow$ twiddle factor

$$X(K) = \sum_{k=0}^{N-1} x(n)\, W_N^{nk}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk}.$$

$X(K) = \text{DFT}[x(n)]$
$x(n) = \text{IDFT}[X(K)]$

** DFT computes N equally spaced frequency samples of the DTFT.

\# Both the indices 'n' and 'k' are ranging from 0 to N-1.

→ The integer 'n' is known as <u>time index</u>, it denotes the time instant.

→ The integer k denotes <u>discrete frequency</u> and called <u>frequency index</u>.

⇒ Twiddle factor is a vector on the unit circle and it represents N equally spaced samples.

$$W_N^{kn}$$ Assume $kn = r$

$$W_N^{r}$$ For $N = 1, 2 \cdots 16$

$$W_8^0 = e^{-0} = 1$$

$$W_8^1 = e^{-\frac{j2\pi}{8}} \qquad W_8^2 = e^{-\frac{j2\pi \times 2}{8}} = e^{-\frac{j\pi}{2}}$$

* In general $W^r = W^{r \pm N} = W^{r \pm 2N} = \cdots$

This is known as periodicity property of twiddle factor.

$$W_8^0 = W_8^8 = W_8^{16} = \cdots = 1$$

$$W_8^1 = W_8^9 = W_8^{17} = \cdots = 1/\sqrt{2} - j/\sqrt{2}$$

* $W_8^1 = 1/\sqrt{2} - j/\sqrt{2}$ and $W_8^5 = -1/\sqrt{2} + j/\sqrt{2}$

Similarly $W_8^2 = -j$ and $W_8^6 = j$

In general, $W^r = -W^{r \pm N/2}$

This is known as symmetry property of twiddle factor.

DFT

→ Frequency analysis of a discrete time signal is performed on a digital signal processor.

→ It converts time domain ~~x(n)~~ sequence $x(n)$ to an equivalent frequency domain $X(\omega)$.

* An aperiodic DTS $x(n)$ with Fourier transform is $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$.

→ Discrete Fourier transform (DFT) :-
(N - Point DFT)

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi K}{N} n} \quad , \quad \text{where } K = 0, 1, 2 \cdots N-1.$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{Kn} \quad \text{where } W_N^{Kn} = e^{-j\frac{2\pi}{N}}.$$

→ Inverse Discrete Fourier transform (IDFT).

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi K}{N} n} \quad ; \quad n = 0, 1, 2 \cdots N-1.$$

* Compute 4-point DFT of causal sequence given by

$$x(n) = \frac{1}{3} \; ; \quad 0 \leq n \leq 2$$
$$= 0 \; ; \quad \text{else}.$$

[from C. Ramesh Babu Page, 162]

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Sol^n

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi k}{N} n}.$$

Here $N=4$.

$$X(K) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi nk}{4}}$$

$$= \sum_{n=0}^{3} x(n)\, e^{-j\frac{\pi nk}{2}}.$$

$$X(K) = x(0)e^{0} + x(1)\, e^{\frac{-j\pi k}{2}} + x(2)\, e^{-j\pi k} + x(3)\, e^{\frac{-j3\pi k}{2}}.$$

The values of $X(k)$ can be evaluated for $K=0,1,2,3$.

$\therefore$ The 4-point DFT sequence of $x(N)$ is

$$X(K) = \left\{1\angle 0, \tfrac{1}{3}\angle -\pi/3, 1/3\angle 0, 1/3\angle \pi/2\right\}$$

Magnitude function $= \left\{1, 1/3, 1/3, 1/3\right\}$

Phase function $= \left\{0, -\pi/3, 0, \pi/2\right\}$

Properties of the DFT

→ **Linearity** If $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

Then $\boxed{a x_1(n) + b x_2(n) \xleftrightarrow[N]{DFT} a X_1(K) + b X_2(k)}$

→ **Periodicity** If $x(n) \xleftrightarrow[N]{DFT} X(K)$

~~then~~ $x(n+N) = x(n)$.

$\boxed{\text{Then } X(K+N) = X(k)}$

→ Circular Shift of Sequence :-

$x_c(n) = x_p(n-k)$

$x_c(n) = x[(n-K), \text{modulo } N]$, $0 \leq n \leq N-1$

$x_c(n) = x[((n-K))_N]$

↳ represented as circular ~~convolution~~ Shift.

Generally, circular ~~convolution~~ Shift is defined

$$x_c(n) = \begin{cases} x_p(n-k) = x[((n-k))_N] & 0 \leq n \leq N-1 \\ 0 & \text{otherwise.} \end{cases}$$

→ * Circular convolution & multiplication of two DFTs.

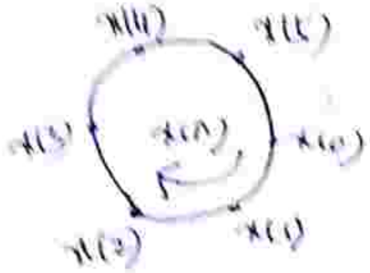
→ Time reversal of a Sequence –

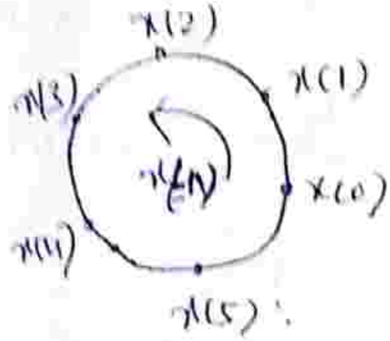
If $x(n) \xleftrightarrow[N]{DFT} X(k)$

then $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N$

$= X(N-k)$

$$\boxed{x(N-n) \xleftrightarrow[N]{DFT} X(N-k)}$$





→ Circular frequency Shift

If $x(n) \xleftrightarrow[N]{DFT} X(k)$

then $x(n)\, e^{\frac{j2\pi ln}{N}} \xleftrightarrow[N]{DFT} X((k-l))_N$.

→ Circular Time Shift

If $x(n) \xleftrightarrow[N]{DFT} X(k)$

then $x((n-l))_N \xleftrightarrow[N]{DFT} X(k)\, e^{\frac{-j2\pi kl}{N}}$.

→ DFT of even & odd sequences :-

- The dft of an even sequence is purely real.
- The dft of an odd sequence is purely imaginary.

→ DFT can be calculated using cosine & sine transforms for even & odd sequences respectively.

* For even sequence

$$\boxed{X(K) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi n K}{N}\right)}$$

* For odd sequence

$$\boxed{X(K) = \sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi n K}{N}\right)}$$

Ex Compute the DFT of sequence defined by $x(n) = (-1)^n$ for (a) $N=3$ (b) $N=4$ (c) $N=\text{even}$ (d) $N=\text{odd}$

C. Ramesh Babu durai Page-174

Sol^n $X(K) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$

$$= \sum_{n=0}^{N-1} x(n) W_N^{KN} = \sum_{n=0}^{N-1} (-1)^n W_N^{KN}$$

$$X(K) = \sum_{n=0}^{N-1} \left[(-1) \times W_N^{K} \right]^n = \sum_{n=0}^{N-1} \left(-W_N^{K} \right)^n$$

Linear convolution Length $= N_1 + N_2 - 1$

* Circular convolution Length $N = \text{Max}(N_1, N_2)$

if $N_1 > N_2$

Methods are used for calculating circular convolution — (i) Concentric circle Method

(ii) Matrix multiplication Method.

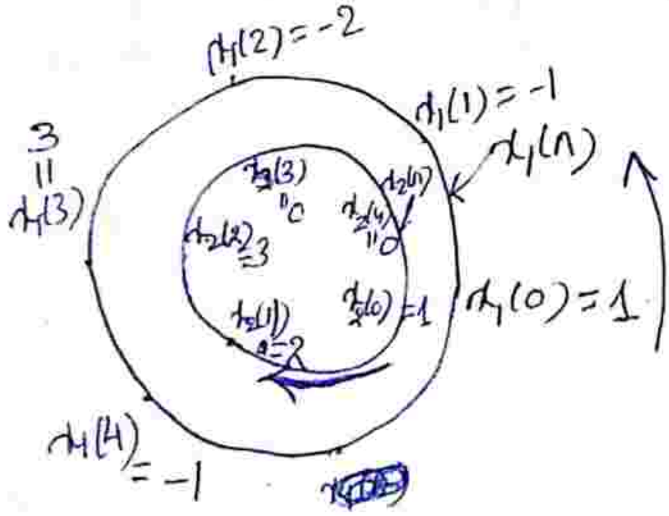
Circular convolution

Q. $x_1(n) = \{1, -1, -2, 3, -1\}$ $x_2(n) = \{1, 2, 3\}$

Soln Needs same length of ~~circular~~ two sequence.

So, We can write $x_1(n) = \{1, -1, -2, 3, -1\}$

$x_2(n) = \{1, 2, 3, 0, 0\}$



$x_1(n)$ is in counterclockwise direction

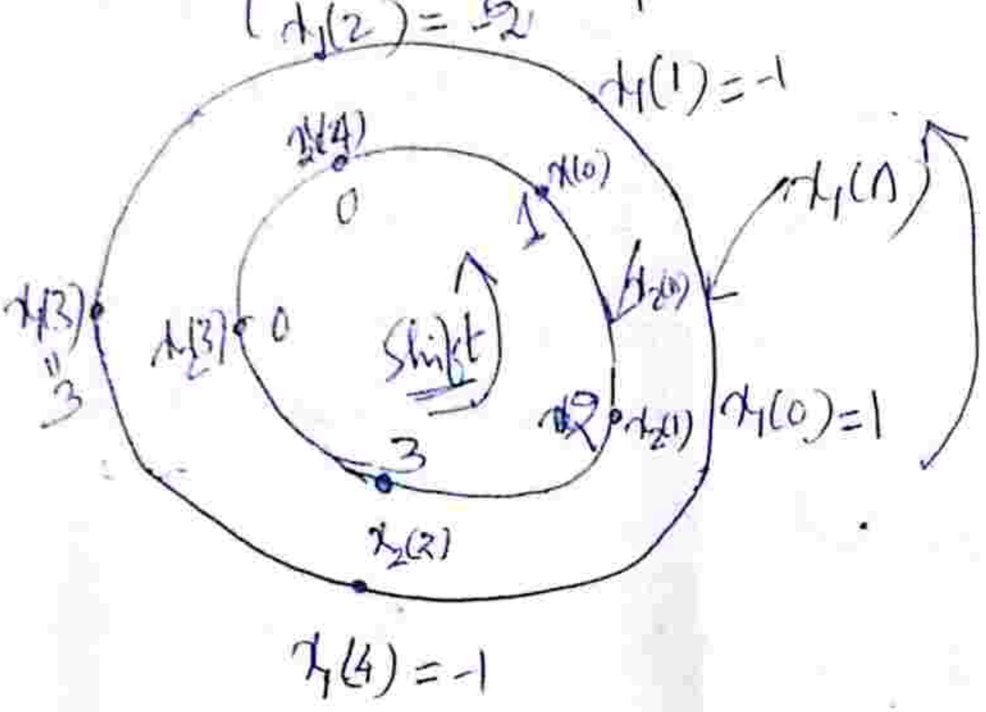
$x_2(n)$ is in clockwise direction

$y(0) = x_1(0)x_2(0) + x_1(1)x_2(4) + x_1(2)x_2(3) + x_1(3)x_2(2) + x_1(4)x_2(1)$.

$y(0) = (1)(1) + (-1)(0) + (-2)(0) + 3(3) + 2(-1)$

$= 1 + 9 - 2 = 8$

→ Rotate the inner circle in anticlockwise direction by one sample, multiply the corresponding samples to obtain $y(1)$.



$y(1) = (1)(2) + (-1)(+1) + (-2)(0) + 3(0) + (-1)3$

$= 2 - 1 - 3 = -2$

Similarly $y(2) = -1$, $y(3) = -4$ & $y(4) = -1$

Result $y(n) = \{8, -2, -1, -4, -1\}$

* Matrix Method

$$\begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

Q Given $x_1(n) = \{1, -1, -2, 3, -1\}$

$x_2(n) = \{1, 2, 3, 0, 0\}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix}}_{x_2(n)} \underbrace{\begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix}}_{x_1(n)} = \underbrace{\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}}_{y(n)}$$

$y(n) = \{8, -2, -1, -4, -1\}$